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From current algebras for p-branes to topological M-theory

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Abstract: In this note we generalize a result by Alekseev and Strobl for the case of p-branes. We show that there is a relation between anomalous free current algebras and "isotropic" involutive subbundles of $T \oplus \wedge^p T^*$ with the Vinogradov bracket, that is a generalization of the Courant bracket. As an application of this construction we go through some interesting examples: topological strings on symplectic manifolds, topological membrane on G_2-manifolds and topological 3-brane on Spin(7) manifolds. We show that these peculiar topological theories are related to the physical (i.e., Nambu-Goto) brane theories in a specific way. These topological brane theories are proposed as microscopic description of topological M/F-theories.

Keywords: p-branes, Topological Field Theories.

Contents

1. Introduction

Recently Hitchin proposed to consider the generalized geometry where the tangent bundle TM is replaced by the tangent plus cotangent bundle $TM \oplus T^*M$. In different context and by different authors it has been pointed out that there is string theory origin of the generalized geometry based on $TM \oplus T^*M$. Indeed many concepts of generalized geometry have their string theory counterpart. Insprired by this relation we would like to make one step further and ask about possible relevant geometric concepts for p-brane theories. In this note we propose that for p-brane theories the relevant geometry is based on $TM \oplus \wedge^p T^*M$ bundle.

The paper consists of two results. First of all we generalize Alekseev-Strobl observation [1] to the case of generic p-brane theory. Namely we associte to anomaly free algebra of p-brane currents an "isotropic" involutive subbundle L of $T \oplus \wedge^p T^*$. This algebra can be regarded as an algebra of first class constraints for some gauge theory. In particular we consider a few interesting examples of such gauge theories, namely topological p-brane theories. We study the compatibility condition between L and riemannian geometry and show that it singles out a very interesting subclass of topological p-brane theories on special class of manifolds. These examples complement the recent discussion of topological M-theory [10, 8, 12, 24, 26] and topological F-theory [2], however at microscopic level. This is our second result.

The structure of the paper is as follows. In section 2 we describe the phase space for p-brane theory which is a simple generalization of the cotangent bundle of loop spaces. In section 3 we associate currents to the sections of $T \oplus \wedge^p T^*$ and calculate the Poisson bracket between them. The calculation gives rise to the Vinogradov bracket on $T \oplus \wedge^p T^*$ (the direct generalization of Courant bracket on $T \oplus T^*$) and a specific anomalous term. The anomaly free subalgebras of the currents can be associated with "isotropic" involutive subbundles of $T \oplus \wedge^p T^*$. We discuss the examples of such subbundles and show that the anomaly free subalgebras of currents can be interpreted as first class constraints of some gauge theory. In section 4 we consider the class of topological p-brane theories which are related to the Nambu-Goto p-branes in a specific way. Actually we obtain the topological strings on symplectic and Kähler manifolds, topological membranes on G_2-manifolds and topological 3-branes on Spin(7)-manifolds. Section 5 presents some comments on the open p-brane theory. In particular we discuss the allowed boundary conditions which preserve the relevant symmetries. In section 6 we summarize and collect some general comments for the future research.

2. Hamiltonian formalism for p-branes

The phase space of closed strings on a manifold M can be identified with the cotangent bundle T^*LM of the loop space $LM = \{X : S^1 \to M\}$. Below we present a straightforward generalization of this construction to the case of generic closed p-brane theory.

Following the logic above for the p-brane world-volume $\Sigma_{p+1} = \Sigma_p \times \mathbb{R}$ the phase space can be identified with the cotangent bundle $T^*\Sigma_p M$ of the space of maps, $\Sigma_p M = \{X : \Sigma_p \to M\}$. Using local coordinates $X^\mu(\sigma)$ and their conjugate momenta $p_\mu(\sigma)$ the standard symplectic form on $T^*\Sigma_p M$ is given by

$$\omega = \int\limits_{\Sigma_p} d^p\sigma \, \delta X^\mu \wedge \delta p_\mu \,, \tag{2.1}$$

where δ is de Rham differential on $T^*\Sigma_p M$. The canonical dimensions of the fields should be chosen such that ω is dimensionless. Namely we choose[1] $dim[X^\mu] = 0$, $dim[\sigma] = 1$ $dim[\partial] = -1$ and $dim[p_\mu] = -p$. The symplectic form (2.1) can be twisted by a closed $(p+2)$-form H, $H \in \Omega^{p+2}(M)$, $dH = 0$, as follows

$$\omega = \int\limits_{\Sigma_p} d^p\sigma \, \left(\delta X^\mu \wedge \delta p_\mu + \frac{1}{2} H_{\mu_1\mu_2\mu_3\ldots\mu_{p+2}} \delta X^{\mu_1} \wedge \delta X^{\mu_2} \epsilon^{\alpha_1\ldots\alpha_p} \partial_{\alpha_1} X^{\mu_3} \cdots \partial_{\alpha_p} X^{\mu_{p+2}} \right), \tag{2.2}$$

where $\epsilon^{\alpha_1\ldots\alpha_p}$ is completely antisymmetric tensor on Σ_p. The symplectic form (2.2) implies the Poisson brackets

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = 0\,, \qquad \{X^\mu(\sigma), p_\nu(\sigma')\} = \delta^\mu_\nu \delta(\sigma - \sigma'), \tag{2.3}$$

$$\{p_\mu(\sigma), p_\nu(\sigma')\} = -H_{\mu\nu\rho_1\ldots\rho_p} \epsilon^{\alpha_1\ldots\alpha_p} \partial_{\alpha_1} X^{\rho_1} \cdots \partial_{\alpha_p} X^{\rho_p} \delta(\sigma - \sigma')\,. \tag{2.4}$$

[1] We work in units where p-brane tension T_p is equal to one. For details see appendix B.

For the symplectic structure (2.2) the transformation

$$X^\mu \rightarrow X^\mu , \qquad p_\mu \rightarrow p_\mu + b_{\mu\nu_1\ldots\nu_p}\epsilon^{\alpha_1\ldots\alpha_p}\partial_{\alpha_1}X^{\nu_1}\cdots\partial_{\alpha_p}X^{\nu_p} \tag{2.5}$$

is canonical if $b \in \Omega^{p+1}(M)$, $db = 0$. There are also canonical transformations which correspond to $Diff(M)$ when X transforms as a coordinate and p as a section of cotangent bundle T^*M. Indeed the group of *local* canonical transformations for $T^*\Sigma_p M$ is a semidirect product of $Diff(M)$ and $\Omega^{p+1}_{closed}(M)$ in analogy with the loop space case [28].

Finally we conclude the discussion of hamiltonian formalism for p-brane theory with the following comment. Typically the symplectic form (2.2) arises from the action

$$S(\gamma) = \int\limits_\gamma (\theta - h), \tag{2.6}$$

where θ is a Liouville form $\omega = \delta\theta$, h is a hamiltonian and γ is a path in $T^*\Sigma_p M$. In order the exponential of this action, $e^{iS(\gamma)}$ to be well-defined we have to impose the intergrality condition on H. Namely we have to require that $[H] \in H^{p+2}(M, \mathbb{Z})$.

3. Current algebra and generalized Dirac structure

In this section we consider the generalization of the idea proposed in [1], where the authors established the relation between 2D anomaly free current algebras and Dirac structures.

Let us consider the currents which are linear in momentum p_μ. If we assume that the currents do not depend on any dimensionful parameter or world-volume metric then the most general form is given by

$$J_\epsilon(v+\omega) = \int\limits_{\Sigma_p} d^p\sigma\ \epsilon\left(v^\mu(X)p_\mu + \omega_{\mu_1\ldots\mu_p}(X)\epsilon^{\alpha_1\ldots\alpha_p}\partial_{\alpha_1}X^{\nu_1}\cdots\partial_{\alpha_p}X^{\nu_p}\right), \tag{3.1}$$

where $v + \omega$ is a section of $T \oplus \wedge^p T^*$ and $\epsilon \in C^\infty(\Sigma_p)$ is a test function. Using the symplectic structure (2.1) we calculate the Poisson bracket of two currents associated to $(v+\omega), (\lambda+s) \in C^\infty(T\oplus\wedge^p T^*)$,

$$\begin{aligned}\{J_{\epsilon_1}(v+\omega), J_{\epsilon_2}(\lambda+s)\} \;=\; & -J_{\epsilon_1\epsilon_2}([[v+\omega,\lambda+s]]) - \\ & -p\int\limits_{\Sigma_p} d^p\sigma\,(\partial_{\alpha_1}\epsilon_1)\epsilon_2(i_v s + i_\lambda\omega)_{\nu_2\ldots\nu_p}\epsilon^{\alpha_1\alpha_2\ldots\alpha_p}\partial_{\alpha_2}X^{\nu_2}\cdots\partial_{\alpha_p}X^{\nu_p},\end{aligned} \tag{3.2}$$

where the bracket $[[\ ,\ ]]$ is defined as follows

$$[[v+\omega,\lambda+s]] = [v,\lambda] + \mathcal{L}_v s - \mathcal{L}_\lambda\omega + d(i_\lambda\omega)\,. \tag{3.3}$$

In (3.3) $[\ ,\ ]$ is the standard Lie bracket on TM and $\mathcal{L}$ is a Lie derivative. Alternatively the result (3.2) can be rewritten as

$$\begin{aligned}\{J_{\epsilon_1}(v+\omega), J_{\epsilon_2}(\lambda+s)\} \;=\; & -J_{\epsilon_1\epsilon_2}([v+\omega,\lambda+s]_c) + \\ & +\frac{p}{2}\int\limits_{\Sigma_p} d^p\sigma\,(\epsilon_1\partial_{\alpha_1}\epsilon_2 - \epsilon_2\partial_{\alpha_1}\epsilon_1)\times \\ & \times(i_v s + i_\lambda\omega)_{\nu_2\ldots\nu_p}\epsilon^{\alpha_1\alpha_2\ldots\alpha_p}\partial_{\alpha_2}X^{\nu_2}\cdots\partial_{\alpha_p}X^{\nu_p}\,,\end{aligned} \tag{3.4}$$

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where the bracket $[\ ,\ ]_c$ is given by

$$[v+\omega, \lambda+s]_c = [v,\lambda] + \mathcal{L}_v s - \mathcal{L}_\lambda \omega - \frac{1}{2} d(i_v s - i_\lambda \omega)\,. \tag{3.5}$$

The bracket $[\ ,\ ]_c$ is just antisymmetrization of the bracket $[[\ ,\ ]]$.

The bracket $[[\ ,\ ]]$ is an example of derived bracket (see [20] for a review) and its antisymmetrization $[\ ,\ ]_c$ is called Vinogradov bracket. One interesting feature is that the bracket $[\ ,\ ]_c$ has non-trivial automorphisms defined by forms [17]. Let $b \in \Omega^{p+1}(M)$ be a closed $(p+1)$-form which defines the vector bundle automorphism e^b of $T \oplus \wedge^p T^*$

$$e^b(v+\omega) \equiv v + \omega + i_v b\,. \tag{3.6}$$

Then the bracket $[\ ,\ ]_c$ satisfies

$$e^b\left([v+\omega,\lambda+s]_c\right) = [e^b(v+\omega), e^b(\lambda+s)]_c\,. \tag{3.7}$$

This non-trivial automorphism of $[\ ,\ ]_c$ corresponds to the canonical transformation (2.5) at the level of Poisson bracket of currents (3.4). If we are interested in the situation when anomalous term is absent in (3.4) and the currents form a closed algebra then we should require the following. Let label the currents by sections of a subbundle $L \subset T \oplus \wedge^p T^*$. In (3.4) the anomalous term is absent if for any $(v+\omega), (\lambda+s) \in C^\infty(L)$

$$\frac{1}{2}(i_v s + i_\lambda \omega) \equiv \langle v+\omega, \lambda+s\rangle = 0\,, \tag{3.8}$$

where $\langle\ ,\ \rangle$ is "pairing" between two sections of $T \oplus \wedge^p T^*$ which is a map $(T \oplus \wedge^p T^*) \times (T \oplus \wedge^p T^*) \ \to \wedge^{p-1} T^*$ where $\wedge^0 T^*$ is understood as $\mathbb{R}$. The bundle automorphism (3.6) preserves this "pairing". We call isotropic any subbundle L which satisfies (3.8). Moreover if we require that our currents form a closed subalgebra then we have to impose that for any two sections $(v+\omega), (\lambda+s) \in C^\infty(L)$ the section $[v+\omega, \lambda+s]_c \in C^\infty(L)$, i.e. the subbundle L is involutive. Indeed the bracket $[\ ,\ ]_c$ restricted to involutive isotropic subbundle of $T \oplus \wedge^p T^*$ is a Lie bracket.[2] Since we could not find the proof of this statement in the literature we present the proof in appendix A as well as other relevant properties of the brackets. The proof is a direct generalization of the proof for $T \oplus T^*$. Thus isotropic involutive subbundle L, as defined above, corresponds to anomaly free algebra of currents

$$\{J_{\epsilon_1}(v+\omega), J_{\epsilon_2}(\lambda+s)\} = -J_{\epsilon_1\epsilon_2}([v+\omega,\lambda+s]_c|_L)\,. \tag{3.9}$$

For the case $p=1$ if L is also maximally isotropic then it is called Dirac structure. In the general situation $p \geq 2$ it is tempting to define a generalized Dirac structure as a maximally isotropic involutive subbundle of $T \oplus \wedge^p T^*$. Although we have to admit that the notion of maximality of isotropic condition (3.8) is not very natural, however see some comments in appendix. For different definitions of generalization of Dirac structure for $T \oplus \wedge^p T^*$ (also called the Dirac-Nambu structure) see [15] and [27].

[2] Indeed L has a structure of the Lie algebroid with the anchor being a natural projection to TM.

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The algebra of currents (3.9) corresponding to involutive isotropic subbundle L can be regarded as an algebra of first class constraints for some gauge theory. In next section we will give a few examples of such theories, namely topological p-branes.

Let us present some examples of isotropic involutive subbundles of $T \oplus \wedge^p T^*$.

Example 1. *Let us fix a $(p+1)$-form, $\phi \in \Omega^{p+1}(M)$ and consider the subbundle $L = \{v + i_v\phi, v \in T\} \subset T \oplus \wedge^p T^*$ which is obviously isotropic*

$$\langle v + i_v\phi, \lambda + i_\lambda\phi\rangle = \frac{1}{2}(i_v i_\lambda \phi + i_\lambda i_v \phi) = 0\,.$$

Next calculate the bracket between two sections

$$[v + i_v\phi, \lambda + i_\lambda\phi]_c = [v, \lambda] + i_{[v,\lambda]}\phi + i_\lambda i_v d\phi\,, \tag{3.10}$$

where we used the property $[\mathcal{L}_v, i_\lambda] = i_{[v,\lambda]}$. The subbundle is involutive if the last term vanishes in (3.10), i.e. $d\phi = 0$. In other words T is involutive isotropic subbundle and $L = e^\phi(T)$, where e^ϕ is the bundle automorphism defined in (3.6) for closed $(p+1)$-form.

The next example is related to the complexification of the bundle $(T \oplus \wedge^p T^*) \otimes \mathbb{C}$.

Example 2. *On complex manifold we can consider the subbundle $L = T_{(1,0)} \oplus (\wedge^p T^*)_{(0,p)}$ of $(T \oplus \wedge^p T^*) \otimes \mathbb{C}$. The sections of L are holomorphic vector fields and antiholomorphic forms (i.e., elements of $\Omega^{(0,p)}(M)$). The subbundle L is obviously isotropic and the bracket of two sections of L is*

$$[v + \omega, \lambda + s]_c = [v, \lambda] + i_v \partial s - i_\lambda \partial\omega$$

which is clearly a section of $T_{(1,0)} \oplus (\wedge^p T^*)_{(0,p)}$. Thus L is an isotropic involutive subbundle.

It is not hard to produce other examples of involutive isotropic subbundles of $T \oplus \wedge^p T^*$, for example based on foliated geometry. In addition we can apply any closed $(p+1)$-form b which defines automorphism (3.6) to an isotropic involutive subbundle L to obtain another isotropic involutive subbundle $e^b(L)$.

So far we calculated the Poisson brackets using (2.1) as symplectic structure. More generally we can calculate the Poisson brackets (3.4) using the twisted symplectic structure (2.2) with $H \in \Omega^{p+2}(M)$, $dH = 0$. In this case the bracket $[\ ,\ ]_c$ in (3.4) gets replaced by its twisted version

$$[v + \omega, \lambda + s]_H = [v + \omega, \lambda + s]_c + i_v i_\lambda H\,. \tag{3.11}$$

All considerations above can be generalized to this case. Thus in particular Example 1 gives rise to isotropic involutive (with respect to $[\ ,\ ]_H$) subbundle if $d\phi = H$.

Finally let us note that the currents (3.1) behave nicely under the diffeomorphisms of Σ_p. Introduce the generator of of $Diff(\Sigma_p)$

$$\mathcal{H}_\alpha[N^\alpha] = \int\limits_{\Sigma_p} d^p\sigma \; N^\alpha \partial_\alpha X^\mu p_\mu \,,$$

where N^α is a text function. The Poisson bracket between generator of $Diff(\Sigma_p)$ and the current (3.1) is

$$\{\mathcal{H}_\alpha[N^\alpha], J_\epsilon(v+\omega)\} = J_{N^\alpha \partial_\alpha \epsilon}(v+\omega) \,,$$

where we assume (2.2) as symplectic structure.

4. Vector cross product and topological branes

In this section we use the construction of involutive isotropic subbundle L given in Example 1 from previous section. For this subbundle we can construct the anomaly free subalgebra of currents (3.9). We interpret these currents as first class constraints for a topological p-brane theory. We impose a specific compatibility of ϕ with a riemannian metric g on M which leads to a certain relation between topological and physical p-brane theories. Indeed all such theories can be classified and there is a finite number of them.

We start by explaining the compatibility condition between the $(p+1)$-form ϕ and a riemannian metric g on M. We all are familiar with the usual vector cross product $\times$ of two vectors in $\mathbb{R}^3$, which satisfies

- $u \times v$ is bilinear and skew symmetric
- $u \times v \perp u, v$; so $(u \times v) \cdot v = 0$ and $(u \times v) \cdot u = 0$
- $(u \times v) \cdot (u \times v) = \det \begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix}$.

The generalization of vector cross product to a riemannian manifold leads to the following definition by Brown and Gray [6]

Definition 3. *On d-dimensional riemannian manifold M with a metric g an p-fold vector cross product is a smooth bundle map*

$$\chi : \wedge^p TM \to TM$$

satisfying

$$\begin{aligned} g(\chi(v_1, ..., v_p), v_i) &= 0 \,, \qquad 1 \leq i \leq p \\ g(\chi(v_1, \ldots, v_p), \chi(v_1, \ldots, v_p)) &= \|v_1 \wedge \cdots \wedge v_p\|^2 \end{aligned}$$

where $\| \cdots \|$ is the induced metric on $\wedge^p TM$.

Equivalently the last property can be rewritten in the following form

$$g(\chi(v_1,\ldots,v_p),\chi(v_1,\ldots,v_p)) = \det(g(v_i,v_j)) = \|v_1\wedge\cdots\wedge v_p\|^2 \,.$$

The first condition in the above definition is equivalent to the following tensor ϕ

$$\phi(v_1,\ldots,v_p,v_{p+1}) = g(\chi(v_1,\ldots,v_p),v_{p+1})$$

being a skew symmetric tensor of degree $p+1$, i.e. $\phi\in\Omega^{p+1}(M)$. Thus in what follows we consider the $(p+1)$-form ϕ which defines the p-fold vector cross product.

Cross product on real spaces were classified by Brown and Gray [6]. The global vector cross products on manifolds were first studied by Gray [13]. They fall into four categories:

1. With $p=d-1$ and ϕ is the volume form of manifold

2. When d is even and $p=1$, we can have a one-fold cross product $J: TM\to TM$. Such a map satisfies $J^2=-1$ and is almost complex structure. The associated 2-form is the Kähler form.

3. The first of two exceptional cases is a 2-fold cross product ($p=2$) on a 7-manifold. Such a structure is called a G_2-structure and the associated 3-form is called a G_2-form.

4. The second exceptional case is 3-fold cross product ($p=3$) on 8-manifold. This is called a Spin(7)-structure and the associated 4-form is called Spin(7)-form.

Notice that there are similarities of this list of real vector cross products with the list of stable forms [16]. Namely the cases (2) and (3) correspond to stability of ϕ. The complexified version of the vector cross product which allows to consider Calabi-Yau manifolds, see [21]. However we will not review the complex version of vector cross product.

Following the discussion from previous section, in particular Example 1, there is a set of topological p-brane theories we can associate to a p-fold vector cross product characterized by $(p+1)$-form ϕ. Consider a subbundle $L=\{v+i_v\phi, v\in T\}$ of $T\oplus\wedge^pT^*$. To the sections of L we can associate the following constraints (currents)

$$J_\mu = p_\mu + \phi_{\mu\nu_1\ldots\nu_p}\epsilon^{\alpha_1\ldots\alpha_p}\partial_{\alpha_1}X^{\nu_1}\cdots\partial_{\alpha_p}X^{\nu_p} = 0\,, \tag{4.1}$$

where we work in local basis ∂_μ. Alternatively we can rewrite the constraints in coordinate free form

$$i_vJ = i_vp + g(\chi(\partial_1X,\ldots,\partial_pX),v) = 0\,, \tag{4.2}$$

where v is a section of TM. The constraints (4.1) are the first class with respect to the symplectic form (2.1) if $d\phi=0$. In the twisted case, when one uses (2.2), the first class condition leads to $d\phi=H$.

Let us now study the compatibility condition between the topological system (4.1) and the Nambu-Goto dynamics. The constraints (4.1) imply the Nambu-Goto costraints (see appendix B)

$$\mathcal{H}_\alpha = p_\mu\partial_\alpha X^\mu = 0 \tag{4.3}$$

$$\mathcal{H} = p_\mu g^{\mu\nu}p_\nu - \det(\partial_\alpha X^\mu g_{\mu\nu}\partial_\beta X^\nu) = 0 \tag{4.4}$$

if and only if ϕ corresponds to vector cross product.[3] Namely

$$\mathcal{H}_\alpha = J_\mu \partial_\alpha X^\mu = 0$$

and

$$p_\mu g^{\mu\nu} p_\nu = \|\partial_1 X \wedge \cdots \wedge \partial_p X\| = \det(g(\partial_\alpha X, \partial_\beta X)) \,,$$

where we have used the second property in the definition of vector cross product. Indeed the Nambu-Goto p-brane theory is decribed by $(p+1)$ constraints (4.3) and (4.4), see appendix B for the details.

We constructed TFTs such that their constraint surface $J_\mu = 0$ lies inside the constraint surface for the standard p-brane theory,

$$J_\mu = 0 \quad \Rightarrow \quad \mathcal{H}_\alpha = 0 \,, \qquad \mathcal{H} = 0 \,.$$

Classically it means that the BRST cohomology of topological branes is subspace of the BRST cohomology of physical brane theory. At quantum level we may speculate that the correlators of observables of topological brane theory are related to subsector of physical brane theory, in analogy with the relation between topological strings and superstrings. However, at the present level of discussion, we cannot elaborate more on the relation between quantum toopological and physical brane theories.

There is an alternative point of view on the relation between the topological p-brane theory and standard p-brane theory (i.e., given by Nambu-Goto (NG) action) on a manifold with a vector cross product structure. Namely the Nambu-Goto action can be thought of as a deformation of the corresponding topological theory. The hamiltonian of Nambu-Goto theory is given by the following expression

$$h_{NG} = \int d^p\sigma \, (N\mathcal{H} + N^\alpha \mathcal{H}_\alpha) \,,$$

where $\mathcal{H}$, $\mathcal{H}_\alpha$ are the constraints (4.3)-(4.4) and N, N^α are lagrangian multipliers. Next assume that ϕ defines a vector cross product with respect to g, so does $-\phi$. We define the currents

$$J^\pm_\mu = p_\mu \pm \phi_{\mu\nu_1\ldots\nu_p} \epsilon^{\alpha_1\ldots\alpha_p} \partial_{\alpha_1} X^{\nu_1} \cdots \partial_{\alpha_p} X^{\nu_p}$$

and rewrite the constraints $\mathcal{H}$, $\mathcal{H}_\alpha$ as follows

$$\mathcal{H} = J^+_\mu g^{\mu\nu} J^-_\nu \qquad \text{and} \qquad \mathcal{H}_\alpha = \partial_\alpha X^\mu J^+_\mu \,,$$

where we used the fact that ϕ is vector cross product with respect to g. As a further preparatory step, we introduce the auxiliary fields $B^\mu_\pm$ and rewrite the NG action as

$$S_{NG} = \int dt\, d^p\sigma \left(p_\mu \dot{X}^\mu - B^\mu_- J^-_\mu - B^\mu_+ J^+_\mu + \frac{1}{N} B^\mu_+ g_{\mu\nu} B^\nu_- - \frac{1}{N} N^\alpha \partial_\alpha X^\mu g_{\mu\nu} B^\nu_- \right) . \qquad (4.5)$$

[3]Indeed previously the cross vector product has been discussed in the context of p-brane instantons for the Nambu-Goto theory [5, 11].

Since the fields B^μ_+ enter linearly we can integrate them and arrive at the standard Nambu-Goto action in the phase space form. Obviously the action (4.5) is not unique and there are other equivalent ways to rewrite it.

For the topological p-brane theory we have two possible (equivalent) hamiltonians

$$h^\pm = \int d^p\sigma \; B^\mu_\pm J^\pm_\mu \,,$$

where $B^\mu_\pm$ are the Lagrange multipliers. In action (4.5) actually both currents $J^\pm_\mu$ enter. However we do not want to introduce two copies of the topological theory and thus one of the two should be fake. This can be easily obtained by considering the action

$$S_{\rm top} = \int dt\, d^p\sigma \left(p_\mu \dot{X}^\mu - B^\mu_- J^-_\mu - B^\mu_+ J^+_\mu - \chi_\mu B^\mu_+ \right), \tag{4.6}$$

where χ_μ is the Lagrange multiplier freezing B^μ_+. Now combining (4.5) and (4.6) it is straightforward to write the Nambu-Goto action as follows

$$S_{NG} = S_{\rm top} - \lambda S_{\rm def} \tag{4.7}$$

where

$$S_{\rm def} = \int dt\, d^p\sigma \left(\frac{1}{N} N^\alpha \partial_\alpha X^\mu g_{\mu\nu} B^\mu_- + \eta_\mu \left(g^{\mu\nu}\chi_\nu + \frac{1}{N} B^\mu_- \right) \right)$$

where η is an additional auxiliary field. In (4.7) at $\lambda = 0$ the theory describes the topological p-brane theory. If λ is non zero then the action (4.7) becomes S_{NG} upon a rescaling of the Lagrange multipliers $N^\alpha \to \lambda^{-1} N^\alpha$. This construction (or its versions) exists only if ϕ corresponds to a vector cross product structure and $d\phi = 0$.

Thus for the list of vector cross product structures given above there is a corresponding list of topological p-brane theories. The first case with ϕ given by the volume structure corresponds to the trivial case when the Nambu-Goto action is itself topological since it describes the embedding of $(d-1)$-branes into a d-dimensional manifold, for details see [4].

We would like to discuss the other three non-trivial cases: topological strings on symplectic manifolds (also on generalized Kähler manifolds), topological membranes on G_2-manifolds and topological 3-branes on Spin(7)-manifolds.

4.1 Topological strings on symplectic manifolds

Case (2) in the list of real vector cross products corresponds to A-model topological strings. 1-fold cross product $J : TM \to TM$ corresponds to an almost complex structure,[4] $J^2 = -1$. The associated 2-form $\omega = gJ$ is the Kähler form. The constraints corresponding to maximally isotropic subbundle $L = \{v + i_v\omega, v \in T\}$ of $T \oplus T^*$ are

$$p_\mu + \omega_{\mu\nu}\partial X^\nu = 0\,. \tag{4.8}$$

They are first class constraints if $d\omega = 0$ and thus the manifold M is symplectic. Indeed this is nothing but A-model topological string theory.

[4]The vector cross product properties read $(gJ)^t = -gJ$ and $J^t g J = g$ which imply $J^2 = -1$.

As far as classical B-model is concern we have to introduce another structure on M. This would correspond to Example 2 in section 3 with $p = 1$. Thus in this case M is a complex manifold with the complex structure J and the constraints are given by

$$p_i = 0\,, \qquad \partial X^{\bar{i}} = 0$$

in complex coordinates. To accomodate both A- and B-models on the same M we have to restrict ourselves to the case of Kähler manifold $(J, g, \omega = gJ)$. In this case we have the following decomposition into holomorphic (antiholomorphic) subbundles

$$(T \oplus T^*) \otimes \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)} \oplus T^{*(1,0)} \oplus T^{*(0,1)}\,. \tag{4.9}$$

There are two interesting sets of complex Dirac structures, first one is $T^{(1,0)} \oplus T^{*(0,1)}$ (or complementary $T^{(0,1)} \oplus T^{*(1,0)}$) and second is $T^{(1,0)} \oplus T^{*(1,0)}$ (or complementary $T^{(0,1)} \oplus T^{*(0,1)}$). Indeed they corresponds to two different generalized complex structures

$$\mathcal{J}_i : (T \oplus T^*) \otimes \mathbb{C} \to (T \oplus T^*) \otimes \mathbb{C}\,, \qquad i = 1, 2$$

such that $\mathcal{J}_i^2 = -1$ and $\Pi^i_\pm = \frac{1}{2}(1 \pm i\mathcal{J}_i)$ project maximally isotropic involutive subbundles of $(T \oplus T^*) \otimes \mathbb{C}$ (for more details see [14]). In the case of Kähler manifolds the corresponding generalized complex structures are

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4.10}$$

which commute and give rise to the following positive metric on $T \oplus T^*$

$$\mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Introducing

$$\Lambda = \begin{pmatrix} i\partial X \\ p \end{pmatrix}$$

as a section of pull-back of tangent and cotangent bundle, $X^*((T \oplus T^*) \otimes \mathbb{C})$ we have four topological string theories given by the set of first class constraints

$$\Pi^i_\pm \Lambda = 0\,. \tag{4.11}$$

Indeed there are only two distinct theories. For the case $T^{(1,0)} \oplus T^{*(0,1)}$ we have $\Pi^2_- \Lambda = 0$, i.e.

$$p_i - i g_{i\bar{j}} \partial X^{\bar{j}} = 0\,, \qquad p_{\bar{i}} + i g_{\bar{i}j} \partial X^j = 0 \tag{4.12}$$

which is A-model topological strings. For the other case $T^{(1,0)} \oplus T^{*(1,0)}$ the constraints are $\Pi^1_- \Lambda = 0$, i.e.

$$p_i = 0\,, \qquad \partial X^{\bar{i}} = 0 \tag{4.13}$$

corresponding to B-model topological strings.[5] Obviously both A- and B-models constraints imply the physical string constraints, $\mathcal{H}_1 = p_\mu \partial X^\mu = 0$ and $\mathcal{H} = p_\mu g^{\mu\nu} p_\nu - \partial X^\mu g_{\mu\nu} \partial X^\nu = 0$. Using the natural pairing $\langle\ ,\ \rangle$ on $T \oplus T^*$ (see (A.5) for $p = 1$) we can rewrite the string constraints as follows

$$-i\mathcal{H}_1 = \langle \Lambda, \Lambda \rangle = 0\,, \qquad 2\mathcal{H} = \langle \Lambda, \mathcal{G}\Lambda \rangle = 0\,. \tag{4.14}$$

Since we have formulated everything in $T \oplus T^*$ covariant language it is not hard to generalize above discussion to the case (twisted) generalized Kähler manifolds as defined in [14]. The generalized Kähler structure is given by two generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ which commute and $\mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2$ defines the positive metric on $T \oplus T^*$.

4.2 Topological membrane on G_2 manifolds

The first exceptional case, namely (3) in the list of real vector cross product structures, corresponds to M being oriented 7-manifold with a global 2-fold cross product structure ($p = 2$). This cross product is defined by riemannian metric g and 3-form Φ which gives rise to a G_2-structure on the manifold.[6] The topological membrane theory on G_2-manifold is defined by the following first class constraints in $T^*\Sigma_2 M$

$$p_\mu + \Phi_{\mu\nu\rho} \epsilon^{\alpha\beta} \partial_\alpha X^\nu \partial_\beta X^\rho = 0\,. \tag{4.15}$$

The algebraic properties of Φ are such that the constraints (4.15) imply the membrane constraints (4.3)-(4.4). $d\Phi = 0$ is equivalent to the fact that (4.15) are first class constraints with respect to the symplectic structure (2.1). We put forward this as the hamiltonian description of recently proposed topological M-theory [10, 8, 12, 24, 26] at microscopic level.

Suppose that G_2-manifold M_7 is of the form $M_7 = M_6 \times S^1$, where M_6 is a six-dimensional manifold with SU(3) structure. Let X^7 be a coordinate along S^1 then Φ can be written as

$$\Phi = \omega \wedge dX^7 + \rho\,, \tag{4.16}$$

where ω is the Kähler 2-form and ρ is the 3-form which defines the almost complex structure on 6-manifold. If ω and ρ do not depend on X^7 then $d\Phi = 0$ implies that $d\omega = 0$ and $d\rho = 0$ on M_6. Membranes on such M_7 can be reduced either to strings on M_6 or to membranes on M_6 depending on the orientation with respect to S^1. If the brane is wrapped along S^1 then we can make a partial gauge fixing $X^7 = \sigma_2/L$ with L being the size of S^1. Then the constraint (4.15) becomes

$$Lp_n + 2L\rho_{nml} \partial_1 X^m \partial_2 X^l + 2\omega_{nm} \partial_1 X^m = 0\,, \qquad p_7 + 2\omega_{nm} \partial_1 X^n \partial_2 X^m = 0\,, \tag{4.17}$$

where $\mu = (n, 7)$. If we want to reinterpret this as a constraint in M_6 we have to redefine the momenta[7] $p_n|_{M_6} \equiv Lp_n$ and restrict our attention only to σ_2 inedpendent configurations

[5] Using the relation $p_\mu = g_{\mu\nu} \dot{X}$ in (4.12) and (4.13) one can recoginize the holomorphic map and constant map conditions over which A- and B-model path integrals are localized correspondently.

[6] In this case the metric g can be expressed in terms of Φ, [19].

[7] See section 2 and appendix for our conventions on the dimensionality of fields.

(e.g., by requiring $\partial_2 X^n = 0$). Assuming this we arrive to the constraint

$$p_n + 2\omega_{nm}\partial_1 X^m = 0 \tag{4.18}$$

which is A-model on M_6. Another possibility corresponds to the case when original membrane does not have excitations along X^7, e.g. X^7 chosen to be a constant. Then in this case the theory on M_6 is membrane theory,[8]

$$p_n + \rho_{nml}\partial_1 X^m \partial_2 X^l = 0\,. \tag{4.19}$$

Since this theory depends on complex moduli it is tempting to call it B-model. Although perturbative B-model is typically defined as a topological string theory there should be a dual formulation in terms of membrane theory. Indeed this option is very natural from geometrical point of view due to the moduli dependence.

4.3 Topological 3-brane on Spin(7) manifolds

The last case in the list of real vector cross products to an oriented 8-manifold M with a global cross product structure with $p = 3$. This cross product gives rise to an associated riemannian metric g and 4-form Ψ. Indeed Ψ is self-dual form $*\Psi = \Psi$, which is called sometime Cayley form and defines Spin(7)-structure on M. The theory is described by the following first class constraints in $T^*\Sigma_3 M$

$$p_\mu + \Psi_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma}\partial_\alpha X^\nu \partial_\beta X^\rho \partial_\gamma X^\sigma = 0\,. \tag{4.20}$$

The algebraic properties of Ψ would follow from the requirement that above constraints imply the 3-brane constraints (4.3)-(4.4). The closure of Ψ is equivalent to the constraints (4.20) being first class with respect to the symplectic structure (2.1). We propose that this topological 3-brane theory is microscopic description of topological F-theory recently discussed in [2].

Let us study two possible reductions of 3-brane topological theory on Spin(7)-manifold down to G_2- and SU(3)-manifolds. As a first case consider Spin(7)-manifold of the form $M_8 = M_7 \times S^1$ with

$$\Psi = dX^8 \wedge \Phi + *\Phi \tag{4.21}$$

where Φ is G_2-structure on M_7 independent on X^8. As result $d\Psi = 0$ implies $d\Phi = 0$ and $d * \Phi = 0$. In analogy with the reduction we discussed in previous subsection a reduction of topological 3-brane theory on M_8 gives a topological membrane theory (with Φ in constraint) theory and topological 3-brane theory (with $*\Phi$ in constraint) on M_7. However topological 3-brane theory cannot be related to 3-brane Nambu-Goto theory in a way described previously.

Following [2] we can consider Spin(7)-manifold $M_8 = M_6 \times T^2$ where M_6 is SU(3)-manifold. Assuming that (X^7, X^8) are coordinates along T^2 the Cayley form is given by

$$\Psi = dX^7 \wedge \rho - dX^8 \wedge \hat{\rho} + dX^7 \wedge dX^8 \wedge \omega + \frac{1}{2}\omega \wedge \omega, \tag{4.22}$$

[8] Using the notion of complex vector cross product we can show that a complex version of the constraints (4.19) implies the membrane Nambu-Goto constraints.

where (ρ, ω) defines SU(3)-structure on M_6, such that $\Omega = \rho + i\hat{\rho}$. We can reduce the topological 3-brane theory given by (4.20) down to M_6. We get a family of topological theories: topological strings (ω), topological 3-branes ($\omega \wedge \omega$) and two topological membranes (for ρ and $-\hat{\rho}$). Since on M_8 topological 3-brane theory is self-dual ($\Psi = *\Psi$), in M_6 we get the duality between topological string (ω) and topological 3-brane ($\omega \wedge \omega$) and another duality between topological membrane theories (ρ and $-\hat{\rho}$). Indeed two first theories can be interpreted as A-model and membrane theories as B-model. This would agree with the expected moduli dependence. Presumably the duality we just discussed is related to proposed S-duality [23].

5. Open p-branes

The open string phase space can be identified with the cotangent bundle T^*PM of the path space $PM = \{X : [0,1] \to M\, X(0) \in D_0, X(1) \in D_1\}$. This construction can be generalized to the case of open p-branes. Assume for the sake of clarity that $\partial\Sigma_p$ consists of one component. For such open p-brane the phase space can be identified with the cotangent bundle $T^*\Sigma_p M_D$ of the space $\Sigma_p M_D = \{X : \Sigma_p \to M, X(\partial\Sigma_p) \subset D\}$ where D is a submanifold of M, $i : D \hookrightarrow M$. To write down the symplectic structure on $T^*\Sigma_p M_D$ we have to require that there exists $B \in \Omega^{p+1}(D)$ such that $dB = i^*H$. Hence the symplectic structure is given by

$$\omega = \int\limits_{\Sigma_p} d^p\sigma \left(\delta X^\mu \wedge \delta p_\mu + \frac{1}{2} H_{\mu_1\mu_2\mu_3\ldots\mu_{p+2}} \delta X^{\mu_1} \wedge \delta X^{\mu_2} \epsilon^{\alpha_1\ldots\alpha_p} \partial_{\alpha_1} X^{\mu_3} \cdots \partial_{\alpha_p} X^{\mu_{p+2}} \right) -$$

$$-\frac{1}{2} \int\limits_{\partial\Sigma_p} d^{p-1}\sigma\, B_{\mu_1\mu_2\mu_3\ldots\mu_{p+1}} \delta X^{\mu_1} \wedge \delta X^{\mu_2} \epsilon^{\alpha_1\ldots\alpha_{p-1}} \partial_{\alpha_1} X^{\mu_3} \cdots \partial_{\alpha_{p-1}} X^{\mu_{p+1}}\,, \qquad (5.1)$$

where the boundary contributions are needed in order ω to be closed, $\delta\omega = 0$. If we require the symplectic form (5.1) to be compatible with the action (2.6) with θ being a Liouville form for $\omega = \delta\theta$ then, in order to the exponent of this action to be well-defined, we have to impose $[(H, B)] \in H^{p+2}(M, D, \mathbb{Z})$, where $H^{p+2}(M, D, \mathbb{Z})$ is an integer relative cohomology group.

Let us introduce a few useful mathematical notions which are generalizations of the ideas from [14] used in the context of $T \oplus T^*$.

Definition 4. *Let M be a manifold with a closed $(p+2)$-form H. Then the pair (D, B) of a submanifold $i : D \hookrightarrow M$ together with a $(p+1)$-form $B \in \Omega^{p+1}(D)$ is a generalized submanifold of (M, H) iff $dB = i^*H$.*

A generalized submanifold (D, B) is exactly the data we need to construct the phase space $T^*\Sigma_p M_D$ together with the symplectic structure (5.1).

Definition 5. *The generalized tangent bundle τ_D^B of the generalized submanifold (D, B) is*

$$\tau_D^B = \{v + \omega \in TD \oplus \wedge^p T^*M|_D : \omega|_D = i_v B\}$$

isotropic subbundle of $(TM \oplus \wedge^p T^*M)|_D$.

If we choose $B = 0$ then $\tau_D^0 = TD \oplus \wedge^p N^*D$, where N^*D is the conormal subbundle of the submanifold D (in other word $N^*D = Ann\, TD \subset T^*M$). The action of the non-trivial automorphism (3.6) of $TM \oplus \wedge^p T^*M$ on generalized submanifolds is given as follows

$$e^b(D, B) = (D, B + b)\,.$$

First consider the simple case when $H = 0$ and $B = 0$. Introducing the currents (3.1) labelled by the section of subbundle L of $TM \oplus \wedge^p T^*M$ we can calculate their Poisson bracket with respect to the symplectic structure (2.1). Thus in the case of boundary the calculation (3.4) is modified

$$\begin{aligned}\{J_{\epsilon_1}(v+\omega), J_{\epsilon_2}(\lambda+s)\} = -J_{\epsilon_1\epsilon_2}([v+\omega,\lambda+s]_c) + \frac{p}{2}\int\limits_{\Sigma_p} d^p\sigma\,(\epsilon_1\partial_{\alpha_1}\epsilon_2 - \epsilon_2\partial_{\alpha_1}\epsilon_1)\times \\ \times(i_v s + i_\lambda \omega)_{\nu_2\ldots\nu_p}\epsilon^{\alpha_1\alpha_2\ldots\alpha_p}\partial_{\alpha_2}X^{\nu_2}\cdots\partial_{\alpha_p}X^{\nu_p} + \\ +\frac{1}{2}\int\limits_{\partial\Sigma_p} d^{p-1}\sigma\,\epsilon_1\epsilon_2(i_\lambda\omega - i_v s)_{\nu_2\ldots\nu_p}\epsilon^{\alpha_2\alpha_3\ldots\alpha_p}\partial_{\alpha_2}X^{\nu_2}\cdots\partial_{\alpha_p}X^{\nu_p}\,.\end{aligned} \tag{5.2}$$

As discussed in section 3 we have to require that L is an isotropic and involutive subbundle of $TM \oplus \wedge^p T^*M$. However now we have to take care of the boundary term in (5.2) to the anomaly. This can be done by requiring that

$$(i_\lambda\omega - i_v s)|_D = 0$$

for any $(v+\omega), (\lambda+s) \in C^\infty(L)$. Moreover we have to insure that the action of the currents (i.e., the transformations they generate) do not change the boundary conditions, $X(\partial\Sigma_p) \subset D$, i.e. v and λ restricted to D should be the sections of TD. We can fulfill these two conditions together with the isotropy condition of L by the following

$$L|_D \;\subset\; TD \oplus \wedge^p N^*D\,,$$

where $L|_D$ is the restriction of subbundle L to the submanifold D. In the general situation if we allow a generalized submanifold (D, B) then the correct condition is

$$L|_D \;\subset\; \tau_D^B\,, \tag{5.3}$$

i.e. $L|_D$ is a subbundle of the generalized tangent bundle of the generalized submanifold (D, B).

6. Conclusions

Let us first of all summarize what we have been finding in the previous sections. We started by studying specific current algebras for extended objects requiring the currents to be linear in the momenta, do not involve any world-volume metric and do not contain any dimensionfull parameter. The current algebras where shown to close under the (twisted or untwisted) Poisson bracket if their structure is parametrized by an "isotropic" involutive subbundle of $T \oplus \wedge^p T^*$. We may interpreted then these currents as first class constraints for topological p-branes theories.

In order to link with the usual Nambu-Goto theory, we required the gauge constraints of the topological theory to imply the ones defining the NG theory itself. Equivalently, we required the topological brane theory to be a topological truncation of the NG one. We have shown that the above requirements, namely the algebra closure and the deformability to the NG theory, correspond to the existence of a *real* cross vector product on the manifold on which the p-brane theory is formulated. This mathematical condition reveals to be quite restrictive leaving with few well defined cases. These, and the induced p-brane topological theories, were listed and analised. One of them was the A-model topological string in six dimensions, which we reconstruct in detail. Through an alternative scheme, we reconstructed the B-model in its usual formulation too. In seven dimensions we encountered membrane theory on G_2 manifolds which upon reduction to six dimensions gave the A-model and a novel membrane theory *naturally* coupled to the complex moduli of the six manifold. Analogous phenomena appeared in the last case of 3-branes on eight dimensional manifolds admitting a Spin(7) structure.

The reduction of topological F-theory from Spin(7)-manifold down to SU(3)-manifold produces a whole set of topological brane theories. Some of them are related to Nambu-Goto theories in the way described above. One is the topological membrane theory which should be a version of the B-model since it couples naturally to the complex moduli. This should be regarded as the nonperturbative completition of the A-model. The whole picture requires further study especially at the quantum level. We believe that the present reduction can be generalized to BV set-up[9] and we hope to come back to this issue in future.

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Note added in proof: after we have finished this work we became aware of two interesting works. In [9] the authors discuss the gauging of sigma model with boundary. Motivated by their example they argue that the notion of isotropic subbundle (3.8) can be extended to

$$\frac{1}{2}(i_v s + i_\lambda \omega) \equiv \langle v + \omega, \lambda + s \rangle = dq\,,$$

where $q \in \Omega^{p-1}(M)$. We find this observation interesting. However it is not clear to us the proper interpretation of this condition within our motivating example.

Also after our paper appeared on the net the different proposal for microscopic description of topological M-theory has been given in [3].

[9] For some discussion of BV formalism applied to open topological membrane see [25, 18].

A. Brackets on $C^\infty(T \oplus \wedge^p T^*)$

In this appendix we collect the relevant properties of the brackets $[[\ ,\ ]]$ and $[\ ,\ ]_c$ defined on the sections of $T \oplus \wedge^p T^*$. The proofs of these properties are similar to those presented in [22], in the context of Courant algebroid.

On smooth sections of $T \oplus \wedge^p T^*$ we can define the bracket

$$[[v + \omega, \lambda + s]] = [v, \lambda] + \mathcal{L}_v s - \mathcal{L}_\lambda \omega + d(i_\lambda \omega)\,, \tag{A.1}$$

which is not skew-symmetric. However it satisfies a kind of Leibniz rule

$$[[A, [[B, C]]\,]] = [[\,[[A, B]], C]] + [[B, [[A, C]]\,]]\,, \tag{A.2}$$

where $A, B, C \in C^\infty(T \oplus \wedge^p T^*)$. The property (A.2) is easily proved from the definition (A.1). In fact the bracket $[[\ ,\ ]]$ makes $C^\infty(T \oplus \wedge^p T^*)$ into a Loday algebra. Next we define a new bracket $[\ ,\ ]_c$ as anitsymmetrization of $[[\ ,\ ]]$

$$[A, B]_c = \frac{1}{2}\left([[A, B]] - [[B, A]]\right). \tag{A.3}$$

The explicite expresion for $[\ ,\ ]_c$ is given by

$$[v + \omega, \lambda + s]_c = [v, \lambda] + \mathcal{L}_v s - \mathcal{L}_\lambda \omega - \frac{1}{2} d(i_v s - i_\lambda \omega)\,. \tag{A.4}$$

Let us introduce "pairing" between two sections of $T \oplus \wedge^p T^*$

$$\langle v + \omega, \lambda + s \rangle = \frac{1}{2}(i_v s + i_\lambda \omega)\,, \tag{A.5}$$

which is a map

$$(T \oplus \wedge^p T^*) \times (T \oplus \wedge^p T^*)\ \rightarrow \wedge^{p-1} T^*\,, \tag{A.6}$$

where $\wedge^0 T^* \equiv \mathbb{R}$. Thus the relation between two brackets (A.1) and (A.4) is as follows

$$[A, B]_c = [[A, B]] - d\langle A, B \rangle\,. \tag{A.7}$$

The bracket $[\ ,\ ]_c$ does not satisfies the Jacobi identity. However it is interesting to examine how it fails to satisfy the Jacobi identity. Let us introduce a trilinear operator, Jacobiator, which measures the failure to satisfy the Jacobi identity

$$Jac(A, B, C) = [\,[A, B]_c, C]_c + [\,[B, C]_c, A]_c + [\,[C, A]_c, B]_c\,. \tag{A.8}$$

We can prove the following property

$$Jac(A, B, C) = d\,(Nij(A, B, C)) \tag{A.9}$$

where Nij is the Nijenhuis operator

$$Nij(A, B, C) = \frac{1}{3}\left(\langle [A, B]_c, C\rangle + \langle [B, C]_c, A\rangle + \langle [C, A]_c, B\rangle\right). \tag{A.10}$$

In order to prove (A.9) we note that

$$[\,[A,B]_c, C]_c = [[\,[[A,B]], C]] - d\langle [A,B]_c, C\rangle \tag{A.11}$$

where we have used (A.7) and the fact that $[[\omega, C]] = 0$ whenever ω is closed form.

As corollary of (A.9) we can establish a few useful theorems. Let us call a subbundle $L \subset T \oplus \wedge^p T^*$ isotropic if for any $A, B \in C^\infty(L)$, $\langle A, B\rangle = 0$, where $\langle\ ,\ \rangle$ is defined by (A.5).

Theorem 6. *If subbundle $L \subset T \oplus \wedge^p T^*$ is isotropic and involutive with respect to bracket $[\ ,\ ]_c$ then $Nij|_L = 0$ and $Jac|_L = 0$.*

Thus the bracket $[\ ,\ ]_c$ restricted to isotropic involutive subbundle of $T \oplus \wedge^p T^*$ is a Lie bracket. If we add the requirement of maximality to isotropic condition then there is the following theorem. By maximal isotropic subbundle L we mean that if the condition

$$\langle v + \omega, \lambda + s\rangle = 0$$

is satisfied for all $(v + \omega) \in C^\infty(L)$ then $(\lambda + s) \in C^\infty(L)$, where $\langle\ ,\ \rangle$ is defined by (A.5).

Theorem 7. *If subbundle $L \subset T \oplus \wedge^p T^*$ is maximally isotropic then the following statements are equivalent:*

- *L is involutive with respect to $[\ ,\ ]_c$*
- *$Jac|_L = 0$*
- *$Nij|_L = 0$.*

For $p = 1$ a maximally isotropic involutive subbundle of $T \oplus T^*$ is called a Dirac structure. Thus for the case $p \geq 2$ we refer to a maximally isotropic involutive subbundle of $T \oplus \wedge^p T^*$ as a generalized Dirac structure.

B. Hamiltonian constaints for p-brane

In this appendix we remind the elements of hamiltonian analysis of the standard p-brane theory. The p-brane theory describes the embedding of a $(p+1)$-dimensional world-volume into a d-dimensional manifold M. The Nambu-Goto action is given by the volume of the embedded $(p+1)$ manifold

$$S = -T_p \int\limits_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\det(g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)}\,, \tag{B.1}$$

where $g_{\mu\nu}$ is the metric with euclidean signature on M and T_p is brane tension. If we put $T_p = 1$ then we choose that $\dim[X] = 0$. In order to carry the hamiltonian analysis we assume $\Sigma_{p+1} = \Sigma_p \times \mathbb{R}$, i.e. $\sigma^a = (\sigma^\alpha, \sigma^0)$ with σ^0 being the evolution parameter.

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Denoting by p_μ the momenta conjugate to X^μ and starting from the Nambu-Goto action (B.1) the constraints can be worked out as [7]

$$\mathcal{H} = g^{\mu\nu} p_\mu p_\nu - \det(q_{\alpha\beta}) \tag{B.2}$$
$$\mathcal{H}_\alpha = p_\mu \partial_\alpha X^\mu \tag{B.3}$$

where

$$q_{\alpha\beta} = g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{B.4}$$

is induced spatial metric on the brane.

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